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Complex Geometry and Dirac Equation

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Complex geometry represents a fundamental ingredient in the formulation of the Dirac equation by the Clifford algebra. The choice of appropriate complex geometries is strictly related to the geometric interpretation of the complex imaginary unit $i = \sqrt{-1}$. We discuss two possibilities which appear in the multivector algebra approach: the σ_{123} and σ_{21} complex geometries. Our formalism provides a set of rules which allows an immediate translation between the complex standard Dirac theory and its version within geometric algebra. The problem concerning a double geometric interpretation for the complex imaginary unit $i = \sqrt{-1}$ is also discussed.

1. INTRODUCTION

In this paper we present a set of rules for passing back and forth between the standard (complex) matrix-based approach to spinors in four dimensions and the geometric algebra formalism. This "translation" is only partial, consistent with the fact that the Hestenes formalism (Hestenes, 1966) provides additional geometrical interpretations. In a pure translation nothing can be predicted which is not already in the original theory. In the new version of Dirac's equation some assumptions appear more natural, some calculations more rapid, and new geometric interpretations for the complex imaginary unit $i = \sqrt{-1}$ appear in the translated version for the first time.

The matrix form of spinor calculus and the vector calculus formulated by Gibbs can be replaced by a single mathematical system, called multivector

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algebra, with which the tasks of theoretical physics can be carried out more efficiently. The multivector algebra derives its power from the fact that both the elements and the operations of the algebra are subject to direct geometric interpretation (Hestenes, 1971). The geometric algebra is surely the most powerful and general language available for the development of mathematical physics (Hestenes and Sobczyk, 1984; Lounesto, 1997). The central result is a representation of the Dirac wave function which reveals a geometric structure, hidden in the conventional formulation (Hestenes and Weingartshofer, 1991).

According to Zeni (1994), "The projection of the Dirac equation into the Pauli algebra eliminates redundancies, simplifying our task to solve this equation, since in the Pauli algebra we work in an eight dimensional space over the real numbers, while in the standard formulation we have to do with a 32-dimensional space over the reals, the space of 4×4 complex matrix $\mathcal{C}_{(4)}$."

Hestenes (1967) states that "The imaginary unit appearing in the Dirac equation and the energy-momentum operator represents the bivector generator of rotations in a space-like plane corresponding to the direction of the electron spin."

We wish to clarify these statements. We agree with fact that in the Pauli algebra (isomorphic to the even part of the space/time algebra $Cl_{1,3}^{+}$) we have only eight real parameters in defining the Dirac spinors, but in defining the most general operator which acts on them, how many real parameters do we need? The imaginary unit i is identified by the bivector $\sigma_{21} \in Cl_{3,0}$. Is this the only opportunity? What about the possibility to identify the complex imaginary unit by the pseudoscalar $\sigma_{123} \in Cl_{3,0}$?

In formulating the Dirac equation by the Pauli algebra we can start from the standard matrix formulation and use the ideal approach to spinors to make a clear translation to the Clifford algebra $Cl_{4,1}$ which is isomorphic to $M_4(\mathcal{C})$. The following step is to reduce the formulation of the Dirac equation to an algebra of smaller dimension, the space-time algebra, $Cl_{1,3}$. Finally, we get a projection of the Dirac equation in the Pauli algebra $Cl_{3,0}$ (Zeni, 1994).

In this paper we shall follow a different approach. We give a set of rules which enables us to immediately write the Dirac equation by using the Pauli algebra. The fundamental ingredients of this translation are the direct identification of the complex imaginary unit $i = \sqrt{-1}$ by elements of the Pauli algebra and the introduction of the concept of "complex" geometry (Rembieliński, 1978; Horwitz and Biedenharn, 1984).

The standard (complex) 4-dimensional spinor

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \equiv \begin{pmatrix} \varphi_1 + i\eta_1 \\ \varphi_2 + i\eta_2 \\ \varphi_3 + i\eta_3 \\ \varphi_4 + i\eta_4 \end{pmatrix}, \qquad \varphi_m, \eta_m \in \mathcal{R}, \quad m = 1, 2, 3, 4 \tag{1}$$

is characterized by eight real parameters, which can be settled in the following eight-dimensional Clifford algebras

$$Cl_{3,0}\ [\sim M_2(\mathcal{C})], \qquad Cl_{1,2}\ [\sim M_2(\mathcal{C})],$$

$$Cl_{0,3}\ [\sim \mathcal{H} \oplus \mathcal{H}], \qquad Cl_{2,1}\ [\sim M_2(\mathcal{R}) \oplus M_2(\mathcal{R})]$$

The natural choice is $Cl_{3,0}$ $[\sim M_2(\mathcal{C})]$, the algebra of the three-dimensional space. Such an algebra allows an immediate geometric interpretation for the Pauli matrices:

	$Cl_{3,0}$
scalar	1
vectors	α_1, σ_2, σ_3
bivectors	$\sigma_2\sigma_1$, $\sigma_2\sigma_3$, $\sigma_3\sigma_1$
trivector	$\sigma_1\sigma_2\sigma_3$

The Pauli algebra can be also represented by the complexified quaternionic ring (De Leo and Rodrigues, 1997, 1998, n.d.):

$\mathcal{H}_c$
1
$\iota\mathcal{I}$, $\iota\mathcal{J}$, $\iota\mathcal{K}$
$\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$
ι

In the following, we prefer to use the vectors $\bar{\sigma} \in Cl_{3,0}$, in order to avoid confusion in the identification of the standard (complex) imaginary unit $i = \sqrt{-1}$ by elements of the Pauli algebra. By identifying the complex imaginary unit $i = \sqrt{-1}$ by elements of $Cl_{3,0}$, we must recognize *two* possibilities

$$i = \sqrt{-1} \rightarrow \sigma_{21} \equiv \sigma_2\sigma_1 \text{ (bivector)}$$

or

$$\sigma_{123} \equiv \sigma_1\sigma_2\sigma_3 \text{ (volume element)}$$

in fact

$$\sigma_{21}^2 = \sigma_{123}^2 = -1$$

Consequently, $\varphi_m + i\eta_m$ can be respectively translated by

$$\varphi_m + \sigma_{21}\eta_m \qquad \text{or} \qquad \varphi_m + \sigma_{123}\eta_m \qquad m = 1, \ldots, 4$$

We propose in this paper a discussion concerning these two different possibilities of translation for the standard complex Dirac theory. These two possibilities are strictly related to the use of *two* different "complex" geometries, namely

$$\text{the} \quad \sigma_{123} \quad \text{and} \quad \sigma_{21} \quad \text{complex geometries}$$

In our formalism the standard physical results are soon reproduced. The possibility of choosing *two* different "complex" geometries in performing our translations will give an embarrassing situation: *two different geometric interpretations* for the complex imaginary unit $i = \sqrt{-1}$, namely

$$\text{bivector} \quad \text{or} \quad \text{volume element}$$

2. PROBABILITY AMPLITUDES AND COMPLEX GEOMETRY

The noncommutativity of the elements of $Cl_{3,0}$ algebra requires that we specify whether our Hilbert space $V_{Cl_{3,0}}$ is to be performed by right or left multiplication of vectors by scalars. We will follow the usual choice and work with a linear vector space under right multiplication by scalars (De Leo and Rodrigues, 1997; Finkelstein *et al.*, 1962, 1963a,b; Adler, 1995; Hestenes, 1975, 1979, 1990; Lounesto, 1986, 1993, 1994; Keller, 1993; Gull *et al.*, 1993). In quantum mechanics, probability amplitudes, rather than probabilities, superimpose, so we must determine what kind of number system can be used for the probability amplitudes $\mathcal{A}$. We need a real modulus function $N(\mathcal{A})$ such that

$$\text{Probability} = [N(\mathcal{A})]^2$$

The first four assumptions on the modulus function are basically technical in nature:

$$\begin{aligned} N(0) &= 0 \\ N(\mathcal{A}) > 0 \qquad &\text{if} \quad \mathcal{A} \neq 0 \\ N(r\mathcal{A}) = |r| N(\mathcal{A}), \qquad &r \text{ real} \\ N(\mathcal{A}_1 + \mathcal{A}_2) &\leq N(\mathcal{A}_1) + N(\mathcal{A}_2) \end{aligned}$$

A final assymption about $N(\mathcal{A})$ is physically motived by imposing the *correspondence principle* in the following form: We require that in the absence of quantum interference effects, probability amplitude superimposition should reduce to probability superimposition. So we have an additional condition on $N(\mathcal{A})$:

$$N(\mathcal{A}_1 \mathcal{A}_2) = N(\mathcal{A}_1) N(\mathcal{A}_2)$$

A remarkable theorem of Albert shows that the only algebras over the reals admitting a modulus functions with the previous properties are the reals $\mathcal{R}$, the complex $\mathscr{C}$, the (real) quaternions $\mathscr{H}$, and the octonions $\mathbb{O}$. The previous properties of the modulus function seem to constrain us to work with *division algebras* (which are finite-dimensional algebras for which $a \neq 0$, $b \neq 0$ imply $ab \neq 0$), in fact

$$\mathscr{A}_1 \neq 0, \qquad \mathscr{A}_2 \neq 0$$

implies

$$N(\mathscr{A}_1\mathscr{A}_2) = N(\mathscr{A}_1)N(\mathscr{A}_2) \neq 0$$

which gives

$$\mathscr{A}_1\mathscr{A}_2 \neq 0$$

A simple example of non-division algebra is provided by the algebra $Cl_{3,0}$ since

$$(1 + \sigma_3)(1 - \sigma_3) = 0$$

guarantees that there are nonzero divisors of zero. So if the probability amplitudes are assumed to be elements of $Cl_{3,0}$, we cannot give a satisfactory probability interpretation. Nevertheless, we know that probability amplitudes are connected to inner products, thus we can overcome the above difficulty by defining an *appropriate* scalar product.

We have four possibilities:

We can define a binary mapping $\langle\Psi|\Phi\rangle$ of $V_{Cl_{3,0}} \times V_{Cl_{3,0}}$ into the scalar(S)/bivectorial(BV) part of $Cl_{3,0}$; we recall that $V_{Cl_{3,0}}$ represents the Hilbert space with elements defined in the Pauli algebra,

$$\langle\Psi|\Phi\rangle_{(S,BV)} = \left(\int d^3x\ \Psi^\dagger\Phi\right)_{(S,BV)}$$

Note that the algebra $(1, \sigma_{21}, \sigma_{23}, \sigma_{31})$ is isomorphic to the quaternionic algebra. Thus, we have the mapping

$$V_{Cl_{3,0}} \times V_{Cl_{3,0}} \rightarrow Cl_{0,2} \sim \mathscr{H}$$

We can also adopt the more restrictive "scalar" projection $\langle\Psi|\Phi\rangle_S$:

$$V_{Cl_{3,0}} \times V_{Cl_{3,0}} \rightarrow Cl_{0,0} \sim \mathscr{R}$$

The last two possibilities are represented by the so-called "complex" geometries

$$\langle\Psi|\Phi\rangle_{(1,\sigma_{21})} \qquad \text{and} \qquad \langle\Psi|\Phi\rangle_{(1,\sigma_{123})}$$

In these case we define the following binary mappings:

$$V_{Cl_{3,0}} \times V_{Cl_{3,0}} \to Cl_{0,1}^{i \to \sigma_{21}} \equiv \mathscr{C}(1, \sigma_{21})$$

$$V_{Cl_{3,0}} \times V_{Cl_{3,0}} \to Cl_{0,1}^{i \to \sigma_{123}} \equiv \mathscr{C}(1, \sigma_{123})$$

In the standard definition of inner product we find the operation of transpose conjugation, $\Psi^\dagger$. How can we translate the transpose conjugation into the geometric algebra formalism?

The Clifford algebra $Cl_{3,0}$ has three involutions similar to complex conjugation. Take an arbitrary element

$$E = E_0 + E_1 + E_2 + E_3 \quad \text{in} \quad Cl_{3,0}$$

written as the sum of a scalar E_0, a vector E_1, a bivector E_2, and a volume element E_3. We introduce the following involutions:

$$E^\bullet = E_0 - E_1 + E_2 - E_3 \qquad \text{grade involution}$$

$$E^* = E_0 - E_1 - E_2 + E_3 \qquad \text{conjugation}$$

$$E^\dagger = E_0 + E_1 - E_2 - E_3 \qquad \text{reversion}$$

The grade involution is an automorphism

$$(E_a E_b)^\bullet = E_a^\bullet E_b^\bullet$$

while the reversion and the conjugation are antiautomorphisms, that is,

$$(E_a E_b)^* = E_b^* E_a^*$$

$$(E_a E_b)^\dagger = E_b^\dagger E_a^\dagger$$

$E^\dagger \equiv E^{\bullet *} \equiv E^{*\bullet}$. We shall show that the reversion can be used to represent the hermitian conjugation.

Let us analyze the products $\Psi^\bullet\Psi$, $\Psi^*\Psi$, and $\Psi^\dagger\Psi$, which involve the three involutions defined within the Clifford algebra $Cl_{3,0}$. We must consider the two possibilities due to the identification of the complex imaginary unit $i = \sqrt{-1}$ by σ_{21} and σ_{123}. Let us perform a real projection of these products,

$$\begin{aligned}(\Psi^\bullet\Psi)_S =_{(i\equiv\sigma_{21})} \{&[(\vartheta_1 + \sigma_{21}\eta_1 + \sigma_{23}\varphi_2 + \sigma_{13}\eta_2)\\ &- \sigma_{123}(\varphi_3 + \sigma_{21}\eta_3 + \sigma_{23}\varphi_4 + \sigma_{13}\eta_4)]\\ &\times [(\varphi_1 + \sigma_{21}\eta_1 + \sigma_{23}\varphi_2 + \sigma_{13}\eta_2)\\ &+ \sigma_{123}(\varphi_3 + \sigma_{21}\eta_3 + \sigma_{23}\varphi_4 + \sigma_{13}\eta_4)]\}_S\\ =\ & \varphi_1^2 - \varphi_2^2 + \varphi_3^2 - \varphi_4^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2\end{aligned}$$

$$=_{(i\equiv\sigma_{123})} \{[(\varphi_1 - \sigma_{21}\varphi_2 + \sigma_{23}\varphi_3 + \sigma_{13}\varphi_4)$$
$$- \sigma_{123}(\eta_3 + \sigma_{21}\eta_2 + \sigma_{23}\eta_3 + \sigma_{13}\eta_4)]$$
$$\times [(\varphi_1 + \sigma_{21}\varphi_2 + \sigma_{23}\varphi_3 + \sigma_{13}\varphi_4)$$
$$+ \sigma_{123}(\eta_1 + \sigma_{21}\eta_2 + \sigma_{23}\eta_3 + \sigma_{13}\eta_4)]\}_S$$
$$= \varphi_1^2 - \varphi_2^2 - \varphi_3^2 - \varphi_4^2 + \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2$$

$$(\Psi^*\Psi)_S =_{(i\equiv\sigma_{21})} \{[(\varphi_1 - \sigma_{21}\eta_1 - \sigma_{23}\varphi_2 - \sigma_{13}\eta_2)$$
$$+ \sigma_{123}(\varphi_3 - \sigma_{21}\eta_3 - \sigma_{23}\varphi_4 - \sigma_{13}\eta_4)]$$
$$\times [(\varphi_1 + \sigma_{21}\eta_1 + \sigma_{23}\varphi_2 + \sigma_{13}\eta_2)$$
$$+ \sigma_{123}(\varphi_3 + \sigma_{21}\eta_3 + \sigma_{23}\varphi_4 + \sigma_{13}\eta_4)]\}_S$$
$$= \varphi_1^2 + \varphi_2^2 - \varphi_3^2 - \varphi_4^2 + \eta_1^2 + \eta_2^2 - \eta_3^2 - \eta_4^2$$
$$=_{(i\equiv\sigma_{123})} \{[(\varphi_1 - \sigma_{21}\varphi_2 - \sigma_{23}\varphi_3 - \sigma_{13}\varphi_4)$$
$$+ \sigma_{123}(\eta_1 - \sigma_{21}\eta_2 - \sigma_{23}\eta_3 - \sigma_{13}\eta_4)]$$
$$\times [(\varphi_1 + \sigma_{21}\varphi_2 + \sigma_{23}\varphi_3 + \sigma_{13}\varphi_4)$$
$$+ \sigma_{123}(\eta_1 + \sigma_{21}\eta_2 + \sigma_{23}\eta_3 + \sigma_{13}\eta_4)]\}_S$$
$$= \varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2$$

$$(\Psi^\dagger\Psi)_S =_{(i\equiv\sigma_{21})} \{[(\varphi_1 - \sigma_{21}\eta_1 - \sigma_{23}\varphi_2 - \sigma_{13}\eta_2)$$
$$- \sigma_{123}(\varphi_3 - \sigma_{21}\eta_3 - \sigma_{23}\varphi_4 - \sigma_{13}\eta_4)]$$
$$\times [(\varphi_1 + \sigma_{21}\eta_1 + \sigma_{23}\varphi_2 + \sigma_{13}\eta_2)$$
$$+ \sigma_{123}(\varphi_3 + \sigma_{21}\eta_3 + \sigma_{23}\varphi_4 + \sigma_{13}\eta_4)]\}_S$$
$$= \varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2 + \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2$$
$$=_{(i\equiv\sigma_{123})} \{[(\varphi_1 - \sigma_{21}\varphi_2 - \sigma_{23}\varphi_3 - \sigma_{13}\varphi_4)$$
$$- \sigma_{123}(\eta_1 - \sigma_{21}\eta_2 - \sigma_{23}\eta_3 - \sigma_{13}\eta_4)]$$
$$\times [(\varphi_1 + \sigma_{21}\varphi_2 + \sigma_{23}\varphi_3 + \sigma_{13}\varphi_4)$$
$$+ \sigma_{123}(\eta_1 + \sigma_{21}\eta_2 + \sigma_{23}\eta_3 + \sigma_{13}\eta_4)]\}_S$$
$$= \varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2 + \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2$$

The first conclusion should be the use of the involution † and the assumption of a "real" geometry. Thus, we should translate

$$(\psi_1^* \ \psi_1^* \ \psi_1^* \ \psi_1^*)\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} \equiv \sum_{m=1}^{4} (\varphi_m^2 + \eta_m^2)$$

by

$$(\Psi^\dagger \Psi)_S$$

Nevertheless, this real projection of inner products gives an *undesired* orthogonality between 1, σ_{21}, and σ_{123}. We know that the complex imaginary unit, $i = \sqrt{-1}$, represents a phase in standard quantum mechanics; thus if we wish to adopt the identifications

$$i = \sqrt{-1} \rightarrow \sigma_{21} \quad \text{or} \quad \sigma_{123}$$

we must abandon the "real" geometry. We have another possibility. Let us rewrite Ψ as follows:

$$\Psi = h_1 + \sigma_{123} h_2, \qquad h_{1,2} \in \mathscr{H}(1, \sigma_{21}, \sigma_{23}, \sigma_{31})$$

The full $\Psi^\dagger\Psi$ product is given by

$$\Psi^\dagger\Psi = (h_1^\dagger - \sigma_{123} h_2^\dagger)(h_1 + \sigma_{123} h_2) = |h_1|^2 + |h_2|^2 + \sigma_{123}(h_1^\dagger h_2 - \text{h.c.})$$

and so

$$\Psi^\dagger\Psi = \text{real part} + \text{vectorial part}$$

Consequently,

$$(\Psi^\dagger\Psi)_S \equiv (\Psi^\dagger\Psi)_{(1,\sigma_{21})} \qquad \sigma_{21}\text{-complex geometry}$$

$$(\Psi^\dagger\Psi)_S \equiv (\Psi^\dagger\Psi)_{(1,\sigma_{123})} \qquad \sigma_{123}\text{-complex geometry}$$

Now, $(1, \sigma_{21})$ and $(1, \sigma_{123})$ do not represent orthogonal states, and our spinor Ψ has four complex orthogonal states, the complex orthogonality degrees of freedom needed to connect a general element of the Pauli algebra to the 4-dimensional Dirac spinor

σ_{21}-complex geometry: $\quad 1, \sigma_1, \sigma_{23}, \sigma_{123}$ $\quad$ orthogonal states

σ_{123}-complex geometry: $\quad 1, \sigma_{21}, \sigma_{23}, \sigma_{31}$ $\quad$ orthogonal states

3. BARRED OPERATORS

We justify the choice of a complex geometry by noting that although there is the possibility to define an anti-self-adjoint operator ∂ with all the properties of a translation operator, *imposing a noncomplex geometry,* there is *no* corresponding self-adjoint operator with all the properties expected for a momentum operator. We can overcome such a difficulty by using a *complex* scalar product and defining as the *appropriate momentum operator*

$$\sigma_{21}\text{-complex geometry:} \qquad \overline{p} \equiv -\overline{\partial}\,\big|\,\sigma_{21}$$

$$\sigma_{123}\text{-complex geometry:} \qquad \overline{p} \equiv -\sigma_{123}\overline{\partial}$$

where $1\,|\,\sigma_{21}$ indicates the right action of the bivector σ_{21}. For σ_{123}, it is not important to distinguish between left and right action because σ_{123} commutes with all the elements in $Cl_{3,0}$. Note that the choice $\overline{p} \equiv -\sigma_{21}\overline{\partial}$ still gives a self-adjoint operator with the standard commutation relations with the coordinates, but such an operator does not commute with the Hamiltonian, which will, in general, be an element of $Cl_{3,0}$. Obviously, in order to write equations that are relativistically covariant, we must treat the space components and time in the same way, hence we are obliged to modify the standard "complex" equations by the following substitutions:

$$\sigma_{21}\text{-complex geometry:} \qquad i\partial^{\mu} \rightarrow \partial^{\mu}\,\big|\,\sigma_{21}$$

$$\sigma_{123}\text{-complex geometry:} \qquad i\partial^{\mu} \rightarrow \sigma_{123}\partial^{\mu}$$

Let us now introduce the complex/linear barred operators. Due to the noncommutative nature of the elements of $Cl_{3,0}$, we must distinguish between left and right actions of σ_{21}, σ_{23}, σ_{31}. Explicitly, we write

$$1\,\big|\,\sigma_{21}, \quad 1\,\big|\,\sigma_{23}, \quad 1\,\big|\,\sigma_{31} \tag{2}$$

to identify the right multiplication of σ_{21}, σ_{23}, σ_{31},

$$(1\,\big|\,\sigma_{21})\Psi \equiv \Psi\sigma_{21}, \qquad (1\,\big|\,\sigma_{23})\Psi \equiv \Psi\sigma_{23}, \qquad (1\,\big|\,\sigma_{31})\Psi \equiv \Psi\sigma_{31}$$

Note that the right action of σ_1, σ_2, σ_3 can be immediately obtained from the operators in (2) by σ_{123} multiplication.

In rewriting the Dirac equation, we need to work with "complex" linear barred operators. Here, we must distinguish between σ_{21}- and σ_{123}-complex geometry. In fact, by working with a σ_{123}-complex geometry it is immediate to prove that

$$1\,\big|\,\sigma_{21}, \quad 1\,\big|\,\sigma_{23}, \quad 1\,\big|\,\sigma_{31}$$

represent σ_{123}-complex/linear operators. In contrast, by working with a σ_{21}-complex geometry we have only one permitted right action, that is,

$$1\,\big|\,\sigma_{21}$$

which represents a σ_{21}-complex/linear operator. Why this counting of parameters? It is simple. In $Cl_{3,0}$ we work with eight real parameters, but the most general linear transformation which can be performed on an element of $Cl_{3,0}$, adopting a σ_{123}-complex geometry, is

$$A + B|\sigma_{21} + C|\sigma_{23} + D|\sigma_{31}, \qquad A, B, C, D \in Cl_{3,0}$$

which contains 32 real parameters, the same number as $M_4(\mathscr{C})$. This explains the possibility of a direct translation between 4×4 complex matrices and the Pauli algebra with σ_{123}-complex geometry

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \leftrightarrow \Psi = \psi_1 + \sigma_{21}\psi_2 + \sigma_{23}\psi_3 + \sigma_{31}\psi_4$$

$$M_4(\mathscr{C}) \leftrightarrow A + B|\sigma_{21} + C|\sigma_{23} + D|\sigma_{31}$$

3.1. σ_{123}-Complex Geometry and Dirac Equation

We have now all the tools to reproduce the Dirac equation within the algebra $Cl_{3,0}$. It is sufficient to translate the standard equation

$$i\Gamma^\mu \partial_\mu \Psi = m\Psi$$

by using the identification of $i = \sqrt{-1}$ with σ_{123} and finding a representation of the Dirac matrices Γ^μ by elements of the Pauli algebra. We observe that the Γ^μ's can be rewritten in terms of elements of $Cl_{3,0}$, by adopting pseudoscalar and left/right action of bivectors. To reproduce the right anticommutation relation which characterizes the Dirac algebra, we perform the following identification:

$$\bar{\bar{\Gamma}} \sim (\sigma_{23}, \sigma_{31}, \sigma_{12})$$

To satisfy the anticommutation relation between Γ^0 and $\bar{\bar{\Gamma}}$, we introduce right actions

$$\Gamma^0 \sim 1|\sigma_{32} \qquad \text{and} \qquad \Gamma^{1,2,3} \sim 1|\sigma_{31}$$

Finally, the hermiticity conditions give

$$\begin{aligned} \Gamma^0 &\equiv \sigma_{123}|\sigma_{32} \\ \Gamma^1 &\equiv \sigma_{123}\sigma_{23}|\sigma_{31} \\ \Gamma^2 &\equiv \sigma_{123}\sigma_{31}|\sigma_{31} \\ \Gamma^3 &\equiv \sigma_{123}\sigma_{12}|\sigma_{31} \end{aligned}$$

The Dirac equation reads

$$\partial_t \Psi \sigma_{23} + \sigma_{23} \partial_x \Psi \sigma_{13} + \sigma_{31} \partial_y \Psi \sigma_{13} + \sigma_{12} \partial_x \Psi \sigma_{13} = m\Psi \qquad (3)$$

Let us multiply the previous equation by the barred operator $\sigma_{123} | \sigma_{23}$,

$$\sigma_{123} \partial_t \Psi \sigma_{23} \sigma_{23} + \sigma_{123} \sigma_{23} \partial_x \Psi \sigma_{13} \sigma_{23} + \sigma_{123} \sigma_{31} \partial_y \Psi \sigma_{13} \sigma_{23} + \sigma_{123} \sigma_{12} \partial_x \Psi \sigma_{13} = m \sigma_{123} \Psi \sigma_{23}$$

By observing that

$$\sigma_{23}^2 = -1, \qquad \sigma_{13} \sigma_{23} = \sigma_{21}, \qquad \sigma_{123}(\sigma_{23}, \sigma_{13}, \sigma_{12}) = -(\sigma_1, \sigma_2, \sigma_3)$$

we find

$$\sigma_{123} \partial_t \Psi + \sigma_1 \partial_x \Psi \sigma_{21} + \sigma_2 \partial_y \Psi \sigma_{21} + \sigma_3 \partial_x \Psi \sigma_{21} = m \Psi \sigma_1 \qquad (4)$$

which represents the *Dirac equation in the Pauli algebra with* σ_{123}*-complex geometry.* This equation is obtained by simple translation, so it reproduces the standard physical contents. We are now ready to give the desired translation rules:

$$\Psi \equiv \begin{pmatrix} \varphi_1 + i\eta_1 \\ \varphi_1 + i\eta_2 \\ \varphi_1 + i\eta_3 \\ \varphi_1 + i\eta_4 \end{pmatrix} \leftrightarrow (\varphi_1 + \sigma_{123}\eta_1) + \sigma_{21}(\varphi_2 + \sigma_{123}\eta_2) + \sigma_{23}(\varphi_3 + \sigma_{123}\eta_3) + \sigma_{31}(\varphi_4 + \sigma_{123}\eta_4)$$

$$\Phi^\dagger \Psi \leftrightarrow (\Phi^\dagger \Psi)_{(1, \sigma_{123})}$$

To give the correspondence rules between 4×4 complex matrices and barred operators, we need to list only the matrix representations for the following barred operators:

$$1, \sigma_{21}, \sigma_{23}, \sigma_{123}, 1|\sigma_{12}, 1|\sigma_{23}$$

All the other operators can be quickly obtained by suitable multiplications of the previous ones. The translation of 1 and σ_{123} is very simple:

$$1 \leftrightarrow \mathbb{1}_{4\times4} \qquad \text{and} \qquad \sigma_{123} \leftrightarrow i\mathbb{1}_{4\times4}$$

The remaining four operators are represented by

$$\sigma_{21} \leftrightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad 1|\sigma_{21} \leftrightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\sigma_{23} \leftrightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad 1|\sigma_{23} \leftrightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

3.2. σ_{21}-Complex Geometry and Dirac Equation

Let us now discuss the possibility to write down the Dirac equation in the Pauli algebra with a σ_{21}-complex geometry. At first glance a problem appears. We do not have the needed parameters in the barred operators to perform a translation. In fact, the most general σ_{21}-complex/linear operator is

$$A + B|\sigma_{21}, \qquad A, B \in Cl_{3,0}$$

and consequently we count only 16 real parameters. We have no hope to set down the 32 real parameters characterizing a generic 4×4 complex matrix. Nevertheless, we observe the possibility to perform the grade involution, *which represents a σ_{21}-complex/linear operation*

$$[\Psi(\alpha + \sigma_{21}\beta)]^{\bullet} = \Psi^{\bullet}(\alpha + \sigma_{21}\beta), \qquad \alpha, \beta \in \mathcal{R}$$

Thanks to this involution we double our real parameters. Let us show the desired translation rules:

$$\Psi \equiv \begin{pmatrix} \varphi_1 + i\eta_1 \\ \varphi_1 + i\eta_2 \\ \varphi_1 + i\eta_3 \\ \varphi_1 + i\eta_4 \end{pmatrix} \leftrightarrow (\varphi_1 + \sigma_{21}\eta_1) + \sigma_{23}(\varphi_2 + \sigma_{21}\eta_2)$$

$$+ \sigma_{123}(\varphi_3 + \sigma_{21}\eta_3) + \sigma_{123}\sigma_{23}(\varphi_4 + \sigma_{21}\eta_4)$$

$$\Phi^{\dagger}\Psi \leftrightarrow (\Phi^{\dagger}\Psi)_{(1,\sigma_{21})}$$

To give the correspondence rules between 4×4 complex matrices and barred operators, we need to list only the matrix representations for the following barred operators:

$$1, \sigma_{21}, \sigma_{23}, \sigma_{123}, 1|\sigma_{21}$$

and give the matrix version of the grade involution. All the other operators can be quickly obtained by suitable combinations of the previous operations. The translation of 1 and $1|\sigma_{21}$ is soon obtained:

$$1 \leftrightarrow \mathbb{1}_{4\times4} \qquad \text{and} \qquad 1|\sigma_{21} \leftrightarrow i\mathbb{1}_{4\times4}$$

The remaining rules are

$$\sigma_{21} \leftrightarrow i\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \sigma_{23} \leftrightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\sigma_{123} \leftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

and finally the grade involution is represented by the following matrix:

$$\bullet\text{-involution} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Let us examine how to translate the Dirac equation

$$i\Gamma^{\mu}\partial_{\mu}\Psi = m\Psi$$

by working with a σ_{21}-complex geometry. First, we modify the previous equation by multiplying it by Γ^0 on the left

$$i\partial_t\Psi + i\Gamma^0\bar{\bar{\Gamma}} \cdot \bar{\partial}\Psi = m\Gamma^0\Psi$$

We observe that [by using the standard representation (Itzykson and Zuber, 1985; Bjorken and Drell, 1964) for the Dirac matrices]

$$\Gamma^0\Psi \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \varphi_1 + i\eta_1 \\ \varphi_2 + i\eta_2 \\ \varphi_3 + i\eta_3 \\ \varphi_4 + i\eta_4 \end{pmatrix}$$
$$\leftrightarrow (\varphi_1 + \sigma_{21}\eta_1) + \sigma_{23}(\varphi_2 + \sigma_{21}\eta_2)$$

$$
\begin{aligned}
&- \sigma_{123}(\varphi_3 + \sigma_{21}\eta_3) - \sigma_{123}\sigma_{23}(\varphi_4 + \sigma_{21}\eta_4) \\
\leftrightarrow &[(\varphi_1 + \sigma_{21}\eta_1) + \sigma_{23}(\varphi_2 + \sigma_{21}\eta_2) \\
&+ \sigma_{123}(\varphi_3 + \sigma_{21}\eta_3) + \sigma_{123}\sigma_{23}(\varphi_4 + \sigma_{21}\eta_4)]^{\bullet}
\end{aligned}
$$

and

$$\Gamma^0\overline{\vec{\Gamma}} \leftrightarrow (\sigma_1, \sigma_2, \sigma_3), \qquad i\mathbb{1}_{4\times 4} \leftrightarrow 1\big|\sigma_{21}$$

Thus, the *translated Dirac equation* reads

$$\partial_t\Psi\sigma_{21} + \sigma_1\partial_x\Psi\sigma_{21} + \sigma_2\partial_y\Psi\sigma_{21} + \sigma_3\partial_z\Psi\sigma_{21} = m\Psi^{\bullet} \tag{5}$$

4. EQUIVALENCE OF COMPLEX GEOMETRIES

In the previous sections, we have performed *two* translated versions of the Dirac equation. Explicitly,

$$\sigma_{123}\text{-complex geometry:} \qquad (\sigma_{123}\partial_t + \nabla\big|\sigma_{21})\Psi = m\Psi\sigma_1 \tag{6}$$

$$\sigma_{21}\text{-complex geometry:} \qquad (\partial_t + \nabla)\Psi\sigma_{21} = m\Psi^{\bullet} \tag{7}$$

where

$$\nabla \equiv \sigma_1\partial_x + \sigma_2\partial_y + \sigma_3\partial_z$$

We discuss in this section the possibility to relate the two equations obtained by imposing different geometries. Let us start by taking the •-involution of equation (6),

$$\sigma_{123}\partial_t\Psi^{\bullet} + \nabla\Psi^{\bullet}\sigma_{21} = m\Psi^{\bullet}\sigma_1 \tag{8}$$

By working with equations (6) and (8) we can reobtain equation (7). To do this, we introduce the idempotents

$$e_{\pm} = \tfrac{1}{2}(1 \pm \sigma_3)$$

and give some relations which will be useful in the following:

$$[e_{\pm}, \sigma_{21}] = 0, \qquad \sigma_1 e_{\pm} = e_{\mp}\sigma_1$$

and

$$\sigma_{123}e_{-} = e_{-}\sigma_{21}, \qquad \sigma_{123}e_{+} = -e_{+}\sigma_{21} \tag{9}$$

Let us multiply equations (6) and (8) from the right respectively by e_- and $\sigma_1 e_+$,

$$\sigma_{123}\partial_t \Psi e_- + \nabla \Psi e_- \sigma_{21} = m\Psi e_+ \sigma_1$$

$$\sigma_{123}\partial_t \Psi^{\bullet} \sigma_1 e_+ - \nabla \Psi^{\bullet} \sigma_1 e_+ \sigma_{21} = m\Psi^{\bullet} e_+$$

By using the relations in (9), we can rewrite the previous equations as follows:

$$(\partial_t + \nabla)\Psi e_- \sigma_{21} = m\Psi e_+ \sigma_1 \tag{10}$$

and

$$(\partial_t + \nabla)\Psi^{\bullet} \sigma_1 e_+ \sigma_{21} = -m\Psi^{\bullet} e_+ \tag{11}$$

By taking the "difference" between these last two equations, we have

$$(\partial_t + \nabla)[\Psi e_- - \Psi^{\bullet} \sigma_1 e_+]\sigma_{21} = m[\Psi e_+ \sigma_1 + \Psi^{\bullet} e_+]$$

By redefining

$$\Phi \equiv \Psi e_- - \Psi^{\bullet} \sigma_1 e_+ \tag{12}$$

and noting that

$$\Phi^{\bullet} = \Psi^{\bullet} e_+ + \Psi \sigma_1 e_- = \Psi^{\bullet} e_+ + \Psi e_+ \sigma_1$$

we find

$$(\partial_t + \nabla)\Phi \sigma_{21} = m\Phi^{\bullet} \tag{13}$$

as anticipated.

We conclude this section by discussing the phase transformations characterizing our equations. It is immediate to show that the phase transformation

$$\Psi \to \Psi e^{\sigma_{123}\alpha}, \qquad \alpha \in \mathcal{R}$$

implies the following transformation on Φ:

$$\Phi \to \Phi e^{\sigma_{21}\alpha}$$

In fact,

$$\begin{aligned} \Phi' &= \Psi e^{\sigma_{123}\alpha} e_- - \Psi^{\bullet} e^{-\sigma_{123}\alpha} \sigma_1 e_+ \\ &= \Psi e_- e^{\sigma_{21}\alpha} - \Psi^{\bullet} \sigma_1 e_+ e^{\sigma_{21}\alpha} \\ &= \Phi e^{\sigma_{21}\alpha} \end{aligned}$$

At this stage, there is no difference in using a σ_{123}- or σ_{21}-complex geometry. So we have an equivalence between σ_{123}- and σ_{21}-complex geometry *within the Pauli algebra.*

5. CONCLUSION

The possibility of using Clifford algebra to describe standard quantum mechanics receives a major boost with the adoption of a complex scalar product (complex geometry). A second important step in this objective of translation is achieved with the introduction of the so-called barred operators, which make it possible to write down a few translation rules which allow one to quickly reproduce in the $Cl_{3,0}$ formalism the standard results of the Dirac theory. All the relations can be manipulated without introducing a matrix representation, greatly simplifying the algebra involved.

In this paper we worked with the Pauli algebra, but we wish to remark that our considerations can be immediately generalized to the spacetime algebra, which represents the natural language for relativistic quantum mechanics.

In the standard literature, the imaginary unit scalar of quantum mechanics is replaced by a bivector. We showed that another possibility is also available, namely the identification of the imaginary unit scalar $i = \sqrt{-1}$ with the pseudoscalar γ_{0123} of the spacetime algebra (σ_{123} in the Pauli algebra). These two geometric interpretations reflect the *two* possible choices in defining a complex geometry within the multivector formalism. At the free-particle level, there is an equivalence in using these two complex scalar products.

We conclude by observing that a possible difference between the σ_{21}- and σ_{123}-complex geometries could appear in the formulation of the Salam–Weinberg model, where the electromagnetic group is obtained by symmetry breaking from the Glashow group $SU(2) \times U(1)$. It appears natural to use

$$\sigma_{21},\ \sigma_{23},\ \sigma_{31},\ \text{and}\ 1|\sigma_{21}$$

as generators of the electroweak group. In this case the right choice should be the adoption of a σ_{21}-complex geometry. After symmetry breaking the remaining electromagnetic group will be identified by the left/right action of the generator σ_{21}. A complete discussion of the Salam–Weinberg model within the multivector formalism will be given in a forthcoming paper (De Leo *et al.*, n.d.).

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